The Ultra Relativistic Maxwell and Proca Equations

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Abstract

We consider the Proca equation which is the Maxwell equation of electromagnetism for a massive particle, in the ultra relativistic limit using Snyder-Sidharth Hamiltonian. There is now an extra parity non-conserving term and we investigate the consequence both for the Proca equation and the Maxwell equation.

1 Introduction

We know that Einstein's relativistic energy momentum relation [13] is given by

$$p_\mu p^\mu = \frac{E^2}{c^2} - p^2 = m^2c^2 \tag{1}$$

or, equivalently

$$E^2 = p^2c^2 + m^2c^4 \tag{2}$$

Now if we take natural units i.e. $\hbar = c = 1$ we can rewrite this equation as

$$E^2 = p^2 + m^2 \tag{3}$$

This equation (3) can be considered only in the case where space time is continuous. But if we consider non-commutative geometry and the Snyder relation of position and momentum [2, 3]

$$[x, p] = \hbar = \hbar[1 + (l/\hbar)^2p^2] \tag{4}$$

where l is the minimum length, we can see that if $l \to 0$ we get back the usual Heisenberg relation of position and momentum. Substitute equation (4) in (3) we get

$$E = (m^2 + p^2[1 + l^2p^2]^{-2})^{1/2} \tag{5}$$

or,

$$E^2 = p^2 + m^2 - \alpha l^2 p^4 \tag{6}$$

This equation is called Snyder-Sidharth Hamiltonian for bosons and for fermions it will be more generally [2]

$$E^2 = p^2 + m^2 + \alpha l^2 p^4 \tag{7}$$

This relation becomes important at high energies, for example at energies which are expected in the LHC (Geneva) [5].

2 Maxwell Equations

Let us consider covariant Maxwell's equations [13] given below

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu \tag{8}$$

This particular equation is in an abridged form with $\mu = 0, 1, 2, 3$ and $\nu = 0, 1, 2, 3$.
Adding $\bar{m}^2 A^\nu$ on both side of equation (8) we get

$$\partial_\mu F^{\mu\nu} + \bar{m}^2 A^\nu = \frac{4\pi}{c} j^\nu + \bar{m}^2 A^\nu \tag{9}$$

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) + \bar{m}^2 A^\nu = \frac{4\pi}{c} j^\nu + \bar{m}^2 A^\nu \tag{10}$$

$$\partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu + \bar{m}^2 A^\nu = \frac{4\pi}{c} j^\nu + \bar{m}^2 A^\nu \tag{11}$$

$$(\partial_\mu \partial^\mu + \bar{m}^2) A^\nu - \partial_\mu \partial^\nu A^\mu = \frac{4\pi}{c} j^\nu + \bar{m}^2 A^\nu \tag{12}$$

$$(\partial_\mu \partial^\mu + \bar{m}^2) A^\nu = \frac{4\pi}{c} j^\nu + \bar{m}^2 A^\nu - \partial_\mu \partial^\nu A^\mu \tag{13}$$

We will now introduce the term $\bar{m}^2 = m^2 + \alpha l^2 p^4$ in equation (13) which means that in the environs of Compton wavelength the term $\alpha l^2 p^4$ which is

going to dominate, is included and we will see how Maxwell's equations gets modified.
From (13) we get,

$$(\partial_\mu \partial^\mu + m^2 + \alpha l^2 p^4) A^\nu = \frac{4\pi}{c} j^\nu + \beta A^\nu - \partial_\mu \partial^\nu A^\mu \tag{14}$$

where $\beta = m^2 + \alpha l^2 p^4$.
Let us for convenience take momentum space by introducing $p_\mu \to \imath \hbar \partial_\mu$. This will help us to derive the modified Einstein's energy momentum relationship. So from equation (14) we, get

$$(p_\mu p^\mu + m^2 + \alpha l^2 p^4) A^\nu = \frac{4\pi}{c} j^\nu + \beta A^\nu - \partial_\mu \partial^\nu A^\mu \tag{15}$$

Since the covariant Maxwell's equations follow Lorenz conditions i.e. $\partial_\mu A^\mu = 0$ therefore the last term in the R.H.S turns out to be zero [11]. So equation (15) becomes

$$(p_\mu p^\mu + m^2 + \alpha l^2 p^4) A^\nu = \frac{4\pi}{c} j_\nu + \beta A^\nu \tag{16}$$

Differentiating equation (16) w.r.t ∂_ν, we get

$$(p_\mu p^\mu + m^2 + \alpha l^2 p^4) \partial_\nu A^\nu = \frac{4\pi}{c} \partial_\nu (j^\nu + \beta A^\nu) \tag{17}$$

Substituting equation (1) in (17) we get,

$$(E^2 - p^2 + m^2 + \alpha l^2 p^4) \partial_\nu A^\nu = \frac{4\pi}{c} \partial_\nu (j^\nu + \beta A^\nu) \tag{18}$$

We can see that equation (18) is similar to conditions in equation (15). So under Lorenz conditions equation (18) can be written as

$$(E^2 - p^2 + m^2 + \alpha l^2 p^4) \partial_\nu A^\nu = 0 \tag{19}$$

which implies that

$$\frac{4\pi}{c} \partial_\nu (j^\nu + \beta A^\nu) = 0 \tag{20}$$

We can see that equation (20) gives some sort of an equation of continuity and also this continuity is different from the usual electromagnetic theory. [10]
Let us now try to find out the solutions in modified Maxwell's equations

under the condition that the LHS of equations (16) LHS turns out to be zero.
Then we get,

$$(E^2 - p^2 + m^2 + \alpha l^2 p^4)A^\nu = 0 \tag{21}$$

We consider plane wave solution for equation (21)

$$A_\mu = exp(\mp \imath kx)\epsilon_\mu(k) \tag{22}$$

Here $\epsilon_\mu(k)$ is the polarization vector and k is the wave vector.

3 Results and Discussions

The general solutions are given by [9,11]

$$k^2 = m^2 + \alpha l^2 p^4, k^\nu \epsilon_\nu(k) = \frac{1}{2} K^{\dot{A}B} \epsilon_{\dot{A}B}(K) = 0 \tag{23}$$

where

$$K_{\dot{A}B} = k^\mu \sigma_{\mu,\dot{A}B} = \begin{bmatrix} k^0 + k^3 & k^1 + \imath k^2 \\ k^1 - \imath k^2 & k^0 - k^3 \end{bmatrix} \text{is} \quad \text{a}\, 2 \times 2\text{matrix}$$

For $m \neq 0$ there are three linearly independent, space like polarization vectors $\epsilon^\mu_\imath(k)$which are usually orthonormalized according to

$$\epsilon^\mu_\imath(k)\epsilon^*_{j,\mu}(k) = \frac{1}{2}\epsilon_{\imath,\dot{A}B}(k)\epsilon^{*\dot{A}B}_j(k) = -\delta_{\imath j}, \imath, j = 0, \pm$$

where

$$\epsilon^{*\dot{A}B}(k) = \epsilon^*_\mu \sigma^{\mu,\dot{A}B}$$

$$\epsilon_{+\dot{A}B}(K) = \sqrt{2} n_{2,\dot{A}} n_{1,B}, \epsilon_{-\dot{A}B}(K) = \sqrt{2} n_{1,\dot{A}} n_{2,B},$$

$$\epsilon_{0,\dot{A}B}(K) = \frac{1}{m + \alpha l p^2}(\kappa_{1,\dot{A}}\kappa_{1,B} - \kappa_{2\dot{A}}\kappa_{2B}) \tag{24}$$

where $n_{\imath\dot{A}}$ are the eigenvectors and $\kappa_{\imath\dot{A}}$ are the wave vectors

$$n_{1,\dot{A}} = \begin{bmatrix} e^{-\imath\theta} cos\frac{\theta}{2} \\ sin\frac{\theta}{2} \end{bmatrix}, n_{1,\dot{A}} = \begin{bmatrix} sin\frac{\theta}{2} \\ -e^{-\imath\theta} cos\frac{\theta}{2} \end{bmatrix} \quad (See Appendix 1)$$

and κ is defined as $\kappa_{\imath,\dot{A}} = \sqrt{\lambda_\imath} n_{\imath,\dot{A}}$ (See appendix 1) where λ_1 is the eigenvalue. Now from the conjugate polarization vectors for equation (24) are given by

$$\epsilon^*_{+\dot{A}B}(K) = \sqrt{2} n_{1,\dot{A}} n_{2,B}, \epsilon^*_{-\dot{A}B}(K) = \sqrt{2} n_{2,\dot{A}} n_{1,B},$$

$$\epsilon^*_{0,\dot{A}B}(K) = \frac{1}{m + \alpha l p^2}(\kappa_{1,\dot{A}}\kappa_{1,B} - \kappa_{2,\dot{A}}\kappa_{2,B}) \tag{25}$$

This $\epsilon^*_(\imath)$ is the conjugate polarization vector of $\epsilon_(\imath)$ given in equation (24) and (25). Now from the above equations we obtain the relations

$$\epsilon_{\imath,\dot{A}B}(k) = \epsilon_{-\imath,B\dot{A}}(k) = \epsilon^*_{-1,\dot{A}B}(k) = \epsilon^*_{1,B\dot{A}}(k) \tag{26}$$

Finally we got the spinors for the modified Proca equation. The three polarization vectors are converted to spinors. We neglect m^2 in equation (9), but not m in equation (24) since as we will see m is small.
If we consider a case where the mass of the particle $m \to 0$ in the equations (24) and (25). Then we can see that we retain all the three equations which could not be possible from the usual Proca equation.

$$\epsilon_{+\dot{A}B}(K) = \sqrt{2} n_{2,\dot{A}} n_{1B}, \epsilon_{-\dot{A}B}(K) = \sqrt{2} n_{1,\dot{A}} n_{2,B},$$

$$\epsilon_{0,\dot{A}B}(K) = \frac{1}{\alpha l p^2}(\kappa_{1,\dot{A}}\kappa_{1,B} - \kappa_{2,\dot{A}}\kappa_{2,B}) \tag{27}$$

And conjugate polarization vector are given by

$$\epsilon^*_{+\dot{A}B}(K) = \sqrt{2} n_{1,\dot{A}} n_{2,B}, \epsilon^*_{-\dot{A}B}(K) = \sqrt{2} n_{w,\dot{A}} n_{1,B},$$

$$\epsilon^*_{0,\dot{A}B}(K) = \frac{1}{\alpha l p^2}(\kappa_{1,\dot{A}}\kappa_{1,B} - \kappa_{2,\dot{A}}\kappa_{2,B}) \tag{28}$$

Now we can see that all three polarized states are retrieved with a term $\alpha l p^2$ which equal to the mass of the photon, let's see what will be the mass for a suitable wavelength.
Now consider the term $\alpha l p^2$ which is nothing but equivalent to mass of photon m_p

$$m_P = \alpha l p^2 \tag{29}$$

$$m_P = \alpha l \frac{(\hbar\omega)^2}{c^2} \tag{30}$$

Here we take the ω to be in the order of X-rays range (i.e.$\omega = 10^{20}s^{-1}$ and $\hbar = 1.054 \times 10^{-27}erg - s$) and we can see below that

$$m_P = l_P \frac{(1.054 \times 10^{-27} \times 10^{20})^2}{(3 \times 10^{10})^2}$$

$$m_P \sim (10^{-65})gms$$

taking

$$l_P \sim 10^{-33}cm$$

This is the same mass deduced by one of the authors (Sidharth) some years ago [6] and recently conformed in the observation by the MAGIC team. It is within the best experimental limit set for the photon mass. [7, 8]

4 Modified Weyl's Equation

These Weyl equations were written for a neutrino initially which was consider to be a mass less particle, but later experimental observation showed that neutrino too have a mass. We will see now how Weyls equation is modified with the inclusion of the extra term.
Now lets us consider Weyl's equation with that extra term in the S-S Hamiltonian

$$[p_0 - \vec{\vec{\sigma}} \cdot p + \alpha l \gamma^5 p^2]\tilde{\phi}_R(p) = 0 \tag{31}$$

and the other is

$$[p_0 + \vec{\vec{\sigma}} \cdot p + \alpha l \gamma^5 p^2]\tilde{\phi}_L(p) = 0 \tag{32}$$

We here we consider the $\vec{\vec{\sigma}}$ to be a 4×4 matrix and $\phi(p)$ are called the Weyl spinors which are also 4×4 , this is due to that extra term. Here after a bit of algebra we will have two equations which represented as Modified Weyl equation.
Now consider the equations (22).Here we will take the complex conjugate of (23) then we get

$$\tilde{\phi}_L^*(p)[p_0^* + \vec{\vec{\sigma}}^* \cdot p + \alpha l (\gamma^5)^* p^2] = 0 \tag{33}$$

Here we multiply (31) and (32). The term $\tilde{\phi}_L^*(p)$ and $\tilde{\phi}_R(p)$ which represents a spinors will have one row matrix and other column matrix multiplying this two will give a scalar quantities which can be take out.

$$[p_0 - \vec{\vec{\sigma}} \cdot p + \alpha l \gamma^5 p^2]\tilde{\phi}_R(p)\tilde{\phi}_L^*(p)[p_0^* + \vec{\vec{\sigma}} \cdot p + \alpha l (\gamma^5)^* p^2] = 0 \tag{34}$$

$$[p_0 - \vec{\vec{\sigma}} \cdot p + \alpha l \gamma^5 p^2][p_0^* + \vec{\vec{\sigma}} \cdot p + \alpha l (\gamma^5)^* p^2] \tilde{\phi}_R(p) \tilde{\phi}_L^*(p) = 0 \tag{35}$$

$$[p_0^2 - p^2 + \alpha l^2 p^4] \tilde{\phi}_R(p) \tilde{\phi}_L^*(p) = 0 \tag{36}$$

$$[E^2 - p^2 + \alpha l^2 p^4] \tilde{\phi}_R(p) \tilde{\phi}_L^*(p) = 0 \tag{37}$$

These represent a scalar bosons and a scalar wave function $\tilde{\phi}_R(p)\tilde{\phi}_L^*(p)$. [11]

5 Conclusion

From modified Maxwell's equation we can see that we are getting a new type of equation of continuity. And also we have got the value of photon mass which turns out to be $m_P \sim (10^{-65} gms$ and also a new scalar boson.

APPENDIX-1[9]

CONVERSION OF FOUR VECTORS INTO SPINORS

Minkowski four-vectors belong to the representation $D(\frac{1}{2},\frac{1}{2} = D(\frac{1}{2},0) \otimes D(0,\frac{1}{2})$ of the Lorentz group. The transition of the usual form of a four-vector $k^\mu = (k^0, \vec{k})$ to the spinor representation is provided by the matrices where σ are the Pauli spin matrices.

$$\sigma^{\mu,\dot{A}B} = (\sigma^0, \sigma), \qquad \sigma^\mu_{\dot{A}B} = (\sigma^0, -\sigma) \qquad (b1)$$

consisting of the two-dimensional unit matrix σ^0 and the Pauli matrices σ^a. Each four vector k^μ is related to a 2×2 matrix

$$K_{\dot{A}B} = k^\mu \partial_{\mu,\dot{A}B} = \begin{bmatrix} k^0 + k^3 & k^1 + \imath k^2 \\ k^1 - \imath k^2 & k^0 - k^3 \end{bmatrix} \qquad (b2)$$

this is Hermitian if the components of k^μ are real. The rules for dotting, undotting, raising, and lowering spinor indices also apply to the indices of the matrices; in particular, we have

$$\sigma^\mu_{\dot{A}B} = \sigma^{\mu,\dot{C}D} \epsilon_{\dot{C}\dot{A}} \epsilon_{DB}, \quad \sigma^\mu_{\dot{A}B} = (\sigma^\mu_{\dot{A}B})^* \qquad (b3)$$

where $\epsilon = \imath \sigma^2$ and $\epsilon_{\dot{C}\dot{A}} = \epsilon_{DB} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

We note that the coefficients of the transpose of a matrix K^T read $K_{B\dot{A}}$ if the ones of K are denoted by $K_{\dot{A}B}$; i.e., transposing a matrix interchanges the spinor indices without moving the position of the overdot. Thus the Hermiticity of the matrices is expressed by

$$\sigma^{\mu,\dot{A}B} = \sigma^{\mu,B\dot{A}}, \quad \sigma^\mu_{\dot{A}B} = \sigma^\mu_{B\dot{A}} \qquad (b4)$$

The σ matrices obeys the important relations

$$\sigma^\mu_{\dot{A}B} \sigma^{\nu,\dot{A}B} = 2g^{\mu\nu}, \quad \sigma^\mu_{\dot{A}B} \sigma^{\nu,\dot{A}\dot{C}} + \sigma^\nu_{\dot{A}B} \sigma^{\mu,\dot{A}\dot{C}} = 2g^{\mu\nu} \delta^{\dot{C}} B$$

$$\sigma^\mu_{\dot{A}B} \sigma_{\mu,\dot{C}D} = 2\epsilon_{\dot{A}\dot{C}} \epsilon_{BD} \qquad (b5)$$

The first of these relations translates the Minkowski inner product of 2 four-vector k^μ and p^μ into

$$2k \cdot p = k_\mu 2g^{\mu\nu} p_\nu = k_\mu \sigma^\mu_{\dot{A}B} \sigma^{\nu,\dot{A}B} p_\nu = K_{\dot{A}B} p^{\dot{A}B}. \qquad (b6)$$

where

$$p^{\dot{A}B} = \sigma^{\sigma\dot{A}B}p_\nu = \begin{bmatrix} p^0+p^3 & p^1+\imath p^2 \\ p^1-\imath p^2 & p^0-p^3 \end{bmatrix}$$

and the second term one implies

$$K_{\dot{A}B}K^{\dot{A}\dot{C}} = k^2\delta_B^{\dot{C}} \qquad (b7)$$

In order to reduce terms involving a matrix $K_{\dot{A}B}$ to spinor products, it is necessary to express $K_{\dot{A}B}$ in terms of spinors. For a real four-vector, the matrix $K_{\dot{A}B}$ is Hermitian and can be decomposed into its eigenvectors $n_{\imath,\dot{A}}(\imath=1,2)$ and eigenvalues $\lambda_\imath$

$$K_{\dot{A}B} = \sum_{\imath=1,2}\lambda_\imath n_{\imath,\dot{A}}n_{\imath,B}, \lambda_{\imath,2} = k^0 \pm |k|,$$

$$n_{1,\dot{A}} = \begin{bmatrix} e^{-\imath\theta}cos\frac{\theta}{2} \\ sin\frac{\theta}{2} \end{bmatrix}, \qquad n_{2,\dot{A}} = \begin{bmatrix} sin\frac{\theta}{2} \\ -e^{-\imath\theta}cos\frac{\theta}{2} \end{bmatrix} \qquad (b8)$$

where θ and ϕ denoted the polar and azimuthal angle of $k = |k|e$,respectively

$$e = \begin{bmatrix} cos\phi sin\theta \\ sin\phi sin\theta \\ cos\theta \end{bmatrix} \qquad (b9)$$

For time like vectors $k^2 > 0$, it is often convenient to include the eigenvalues of λ_1 in the normalization of the eigenvectors resulting in

$$\kappa_{\dot{A}B} = \sum_{\imath=1,2}\kappa_{\imath,\dot{A}}\kappa_{\imath,B}, \qquad \kappa_{\imath,\dot{A}} = \sqrt{\lambda_\imath}n_{\imath,\dot{A}} \qquad (b10)$$

The phases $n_{1,\dot{A}}$ are chosen such that the orthonormality relations read

$$< n_\imath n_\imath >= 0, \qquad < n_2 n_1 >=< n_1 n_2 >= +1 \qquad (b11)$$

The special case of a light like vector ($k^2 = 0$) is of particular importance. In this case the eigenvalue λ_2 of Equation ($b8$) vanishes, and the matrix $K_{\dot{A}B}$ factorizes into a single product of two spinors

$$K_{\dot{A}B} = k^\mu\sigma_{\mu,\dot{A}B} = k_{\dot{A}}k_B, \qquad (b12)$$

$$k_{\dot{A}} = \sqrt{2k^0}n_{1,\dot{A}} = \sqrt{2k^0}\begin{bmatrix} e^{-\imath\phi}cos\frac{\theta}{2} \\ sin\frac{\theta}{2} \end{bmatrix} \qquad (b12)$$

In this context, $k_{\dot{A}}$ is called a momentum spinor.
Finally, we remark that the decomposition $(b8)$ is a very convenient, but not unique, possibility to express a fourvector k^μ with $k^2 \neq 0$ in terms of WvdW spinors. Any splitting of k^μ into two lightlike four-vectors yields decomposition into spinors, since light like vectors factorize, as seen above.
For instance, choosing an arbitrary light like fourvector $a^\mu (a^2 \neq 0)$ with $a \cdot k > 0$ and defining

$$a = \frac{k^2}{2a \cdot k}, \qquad b^\mu = k^\mu - a a^\mu \qquad (b13)$$

yields a possible decomposition $k^\mu = a a^\mu + b^\mu$. In terms of WvdW spinor, this correspond to an arbitrarily chosen spinor $a_{\dot{A}}$ with $K_{\dot{C}D} a^{\dot{C}} a^D > 0$, leading to the decomposition

$$K_{\dot{A}B} = a a_{\dot{A}} a_B + b_{\dot{A}} b_B$$

with

$$b_{\dot{A}} = -\frac{K_{B\dot{A}} a^B}{\sqrt{K_{\dot{C}D} a^{\dot{C}} a^D}}, \quad a = \frac{k^2}{k_{\dot{C}D} a^{\dot{C}} a^D} \qquad (b14)$$

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